Benefits of Coding on Age of Information in Broadcast Networks

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***Abstract*—Age of Information (AoI) is studied in two-user broadcast networks with feedback, and lower and upper bounds are derived on the expected weighted sum AoI of the users. In particular, a class of simple coding actions is considered and within this class, randomized and deterministic policies are devised. Explicit conditions are found for symmetric dependent channels under which coded randomized policies strictly outperform the corresponding uncoded policies. Similar behaviour is shown numerically for deterministic policies.**

I. Introduction

Sending status updates in a timely manner has significant importance in the Internet of Things (IoT) applications. To measure the timeliness of information at a remote system, the concept of Age of Information (AoI) was introduced in [1]. AoI measures, at the receiving side, how much time has passed since the generation time of the latest received packet. In [2], a single source and server setup was considered under First-Come First-Serve (FCFS) queue management and it was shown that there is an optimal update rate that minimizes time-average AoI. Further extensions to networks of multiple sources and servers with and without packet management were studied in [3], [4]. The role of packet loss was investigated in [5]–[7] and various coding schemes were designed for the purpose of minimizing AoI. Recently, [7] proved that when the source alphabet and channel input alphabet have the same size, with queue management, a Last-Come First-Serve (LCFS) with no buffer policy is optimal. In other words, source/channel coding is not beneficial in this scenario. This is in contrast to FCFS M/G/1 queues where there is an optimal blocklength for channel coding to minimize the average age and average peak age of information [8]. In multicast networks, [9] studies the impact of network coding on the age of information considering parameters such as the number of packets in a coding block as well as the size of the underlying finite field.

We aim to shed light on the interplay between AoI and (channel/network) coding in the context of broadcast packet erasure channels (BPEC) with feedback. In contrast to the existing literature on BPECs, in this work, we seek *efficiency in terms of age as opposed to rate*. The underlying challenge is as follows. On the one hand, the highest rate of communication in BPECs can be attained when coding is employed across packets of different users [10]. A higher rate effectively corresponds to a smaller delay (both in the sense that the queues get emptied faster and in the sense that fewer uses of the network are needed in total to transmit a fixed number of information bits). On the other hand, to achieve high rates with coding, we have to incur delay by waiting for the arrival of other packets for the purpose of coding as well as prioritizing their transmission. So it is not clear a-priori when coding is beneficial. We will devise scheduling policies that *schedule different coding actions*, as opposed to schemes that schedule users, and show the benefit of coding, in terms of age, over uncoded schemes such as those proposed in [11].

We consider a discrete time model as in [11], devise coding algorithms, and study the expected weighted sum of AoI (EWSAoI) at the users. The first contribution of the paper is a general lower bound on the achievable EWSAoI. As opposed to previous lower bounds (e.g. [11]) that hold only in the class of traditional scheduling algorithms, the new lower bound is valid for any coding scheme. The second contribution of the paper is that we devise scheduling policies on coding actions to serve the users. To this end, we restrict attention to a simple class of coding algorithms consisting of three actions: uncoded transmission for user 1, uncoded transmission for user 2, and coded transmission of overheard packets for both users. We devise and analyze EWSAoI or an upper bound on it for (i) stationary randomized policies and (ii) deterministic Max-Weight (MW) policies. In the class of randomized policies, for symmetric channels, we find conditions under which coded policies perform strictly better than their corresponding uncoded policies. For MW policies, we find an upper bound on the achievable EWSAoI. Furthermore, we numerically compare the performance of coded and uncoded MW policies and show the gains of coding.

II. System model

We consider a model where time is slotted. At the beginning of every time slot, new packets are sampled for each user. Transmission occurs on a noisy network which we model by a two-user broadcast packet erasure channel with feedback, see Fig. 1 (left). In each time slot k, the input of the channel is the packet $X(k)$ and the output at user i is:

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$$Y_i(k) = \begin{cases} X(k) & \text{if } Z_i(k) = 1 \\ \Delta & \text{otherwise} \end{cases}$$

where $\{Z_i(k)\}_{k=1}^{\infty}$ is an iid Bernoulli process with probability $1-\epsilon_i$ modeling erasure at user $i \in \{1,2\}$ in time slot $k \in \{1,2,\ldots\}$ and Δ is the symbol denoting erasure. Due to the available feedback, the encoder has the knowledge of $\{(Y_1(\ell), Y_2(\ell))\}_{\ell=1}^{k}$ at time $k+1$.

Note that the pairs $\{(Z_1(k), Z_2(k))\}_k$ are independent across time (over $k = 1, \ldots$) but potentially correlated across (Z_1, Z_2). Define $\epsilon_1, \epsilon_2, \epsilon_{12}$ as $\epsilon_1 := \Pr(Z_1 = 0)$, $\epsilon_2 := \Pr(Z_2 = 0)$, $\epsilon_{12} := \Pr(Z_1 = 0, Z_2 = 0)$. In addition, $\Pr(Z_1 = 1, Z_2 = 0) = \epsilon_2 - \epsilon_{12}$, $\Pr(Z_1 = 0, Z_2 = 1) = \epsilon_1 - \epsilon_{12}$. The statistics of (Z_1, Z_2) is assumed fixed and given and can be characterized by $(\epsilon_1, \epsilon_2, \epsilon_{12})$.

We consider the age of information, defined below, as the performance metric of our communication design.

Definition 1 (Age of Information [1])**.** *Consider a source-destination pair. Let $\{t_k\}_k$ be the times at which packets are generated and $\{t'_k\}_k$ be the times at which packets are received at the destination. At any time ξ, denote $N(\xi) = \max\{k | t'_k \leq \xi\}$, and $u(\xi) = t_{N(\xi)}$. The Age of Information (AoI) at the destination is $h(t) = t - u(t)$.*

For any communiction policy π, denote the resulting AoI at user i, $i = 1,2$, by $h_i^{\pi}(k)$, $k = 1, 2, \ldots$. We seek to design policies that minimize the following expected weighted sum AoI at the users:

$$\mathbb{E}\left[\frac{1}{2T}\sum_{k=1}^{T}\sum_{i=1}^{2}\alpha_i h_i^{\pi}(k)\Big|\vec{h}_0\right] \tag{1}$$

where $\vec{h}_0$ denotes the initial age pair $(h_1(1), h_2(1))$, and α_1 and α_2 are weights associated to users 1 and 2, respectively. We assume $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 = 1$. Hence the minimum age is given by the following optimization problem (where we have we omitted $\vec{h}_0$ for notational simplicity):

$$\min_{\pi \in \Pi} E[J_T^{\pi}], \quad \text{where} \quad J_T^{\pi} = \frac{1}{2T}\sum_{k=1}^{T}\sum_{i=1}^{2}\alpha_i h_i^{\pi}(k). \tag{2}$$

III. The Lower Bound

We prove a lower bound on EWSAoI as stated below. For $i = \{1,2\}$, we use the notation $-i$ as short for $\{1,2\}\backslash i$.

Theorem 1. *For any communication policy, we have*

$$E[J_T^{\pi}] \geq \frac{1}{4}\left(\frac{(\sum_{i=1}^{2}\sqrt{\alpha_i(2-\epsilon_{12}-\epsilon_{-i})})^2}{(1-\epsilon_{12})(2-\epsilon_1-\epsilon_2)} + 1\right).$$

Proof Outline. Consider a general sample path associated with a communication policy and a finite time-horizon T. For this sample path, let $N_i(T)$ be the total number of packets delivered to user i up to and including time slot T, and $I_i(m)$ be the number of time slots between the $(m-1)$th and mth deliveries to user i, i.e., the inter delivery times of user i. Denote the age of user i after delivery of the mth packet by $D_i(m)$ and let L_i be the number of remaining time slots after the last packet delivery to the same user. With this notation, the time-horizon can be written as $T = \sum_{m=1}^{N_i(T)} I_i(m) + L_i$.

Next, consider the sum of the instantaneous ages in the interval corresponding to $I_i(m)$, $m \geq 2$, denoted by $\Delta_i[m]$. As shown in Fig. 2, $\Delta_i[m]$ is equal to the area underneath the age curve in the corresponding interval minus $I_i(m)$ small triangle areas,

$$\begin{aligned} \Delta_i[m] = &\sum_{k \text{ in between delivery of } m-1^{th} \text{ and } m^{\text{th}} \text{ packets}} h_i[k] \\ =&\frac{(D_i(m-1)+I_i(m))^2}{2} - \frac{(D_i(m-1))^2}{2} - \frac{I_i(m)}{2} \\ =&\frac{I_i^2(m)}{2} + D_i(m-1)I_i(m) - \frac{I_i(m)}{2}. \end{aligned} \tag{3}$$

The expected weighted sum AoI as defined in (1) can be re-written in terms of $\Delta_i(m)$'s:

$$\begin{aligned} J_T^{\pi} =&\frac{1}{2T}\sum_{k=1}^{T}\sum_{i=1}^{2}\alpha_i h_i(k) \\ =&\frac{1}{2}\sum_{i=1}^{2}\frac{\alpha_i}{T}\left(\sum_{m=1}^{N_i(T)}\Delta_i(m) + \frac{1}{2}L_i^2 + D_i(N_i(T))L_i - \frac{1}{2}L_i\right). \end{aligned} \tag{4}$$

Note in (3) that $D_i(m-1) \geq 1$. Substituting $D_i(m-1) = 1$ in (3), we recover [11, Eqn. (9)]. Using similar steps as [11, Eqn. (9) - (14)], we have

$$\lim_{T\to\infty} J_T^{\pi} \geq \frac{1}{4}\sum_{i=1}^{2}\alpha_i\frac{T}{N_i(T)} + \frac{1}{4}. \tag{5}$$

We now depart from [11] by allowing for any general class of coding and scheduling schemes. Recall that $N_i(T)$ is the total number of packets received by user i, $i = 1,2$. In the limit of $T \to \infty$, $\frac{N_i(T)}{T}$ is the throughput of user i. We furthermore know the capacity of two-user BPECs from [10]. In particular, any non-negative rate pair (R_1, R_2) is achievable if and only if it satisfies the following conditions:

$$\frac{R_1}{1-\epsilon_1} + \frac{R_2}{1-\epsilon_{12}} \leq 1, \quad \frac{R_1}{1-\epsilon_{12}} + \frac{R_2}{1-\epsilon_2} \leq 1. \tag{6}$$

Therefore, we have

$$\lim_{T\to\infty}\frac{\frac{N_1(T)}{T}}{1-\epsilon_1} + \frac{\frac{N_2(T)}{T}}{1-\epsilon_{12}} \leq 1 \tag{7}$$

$$\lim_{T\to\infty}\frac{\frac{N_1(T)}{T}}{1-\epsilon_{12}} + \frac{\frac{N_2(T)}{T}}{1-\epsilon_2} \leq 1. \tag{8}$$

Using (7) and (8) along with Cauchy-Schwarz inequality, we continue from (5) and derive

$$\begin{aligned} \lim_{T\to\infty} J_T^{\pi} \geq& \frac{1}{4}\lim_{T\to\infty}\left(\frac{\alpha_1}{N_1(T)/T} + \frac{\alpha_2}{N_2(T)/T}\right) + \frac{1}{4} \\ \geq& \frac{1}{4}\left(\frac{(\sum_{i=1}^{2}\sqrt{\alpha_i(2-\epsilon_{12}-\epsilon_{-i})})^2}{(1-\epsilon_{12})(2-\epsilon_1-\epsilon_2)} + 1\right). \end{aligned}$$

Details of the proof can be found in [12]. □

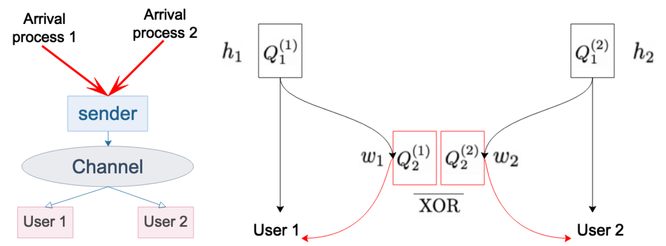

Fig. 1: Broadcast packet erasure channel (left), Encoder as a virtual network of queues (right)

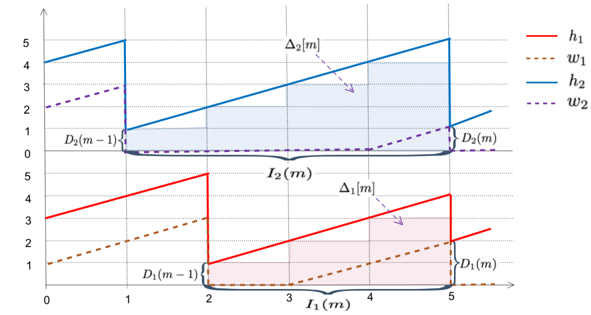

Fig. 2: a sample path of the channel state (h_1, h_2, w_1, w_2) which initial state $(h_1, h_2, w_1, w_2) = (3, 4, 1, 2)$

IV. A Network Coding Scheme

Traditional network strategies schedule transmissions to the users [11]. It is, however, known that overheard packets can act as side information, create coding opportunities, and increase the rate of communication. In particular, [10] has proposed encoding algorithms that track overheard packets in a virtual network of queues and send XOR of overheard packets when possible. We follow this approach and model the encoder by a virtual network of queues as shown in Fig. 1 (right). Let $Q_1^{(i)}$ denote the (virtual) queue of incoming packets for user i and $Q_2^{(i)}$ the (virtual) queue of packets that are intended for user i but are received only by the other user. The packets in $Q_2^{(i)}$ are not received at their intended receivers, but are received at the other receiver and act as side information for it – this can be exploited in the code design at the encoder. In particular, the encoder can XOR packets in $Q_2^{(1)}$ with $Q_2^{(2)}$ and form more efficient coded packets for transmission. In this model, we assume a finite buffer size 1 for every (virtual) queue because it is always more efficient (in terms of age) to transmit the last generated packet and disregard the rest with packet management [12, Lemma 1].

We consider a simple class of coding algorithms that consists of three actions, including a network coding action. In each time slot k, the encoder decides between the following three actions, denoted by $A(k) \in \{1, 2, 3\}$ and defined below:

- $A(k) = 1$: a packet is transmitted from $Q_1^{(1)}$;
- $A(k) = 2$: a packet is transmitted from $Q_1^{(2)}$;
- $A(k) = 3$: a coded packet is transmitted from $Q_2^{(1)}$, $Q_2^{(2)}$.

Let $h_i^{\pi}(k)$ be the positive real number that represents age at user i at time k. For simplicity, we drop the superscript π. The age h_i increases linearly in time when there is no delivery of packets to user i and drops with every delivery to a value that represents how old the received packet is. To capture the evolution of h_i in the class of 3−action algorithms described, we proceed as follows.

First, we define $w_i(k)$ as the (current) age of information at $Q_2^{(i)}$ in slot k. More specifically, if at time k there is a packet p in $Q_2^{(i)}$ with generation time t_p, then the age $w_i(k)$ is equal to $w(k) = k - t_p$. We define $w_i(k)$ to be zero if there is no packet in $Q_2^{(i)}$. Below, we describe the evolution of $w_i(k)$.

Suppose at time k there is a packet p in $Q_2^{(i)}$. At time $k+1$, if packet p is replaced by a new packet l, the age becomes $w_i(k+1) = 1$ (recall that packets are sampled at every time slot and they replace old packets. Hence the packet ℓ was generated one time slot before its move to $Q_2^{(i)}$). If packet p is successfully delivered at user i, then it is removed from $Q_2^{(i)}$ and $w_i(k+1) = 0$. Moreover, once a packet is delivered successfully at user i from $Q_1^{(i)}$, then the existing packet in $Q_2^{(i)}$ (which is necessarily older) becomes obsolete and hence we remove it from $Q_2^{(i)}$ and the age $w_i(k+1)$ is set to 0. Thus, the recursion of $w_i(k)$ can be written as follows:

$$w_i(k+1) = \begin{cases} 0 \text{ if } A(k) \in \{i, 3\},\ Z_i(k) = 1 \\ 1 \text{ if } A(k) = i,\ (Z_i(k), Z_{-i}(k)) = (0, 1) \\ (w_i(k) + 1) \cdot 1_{\{w_i(k) > 0\}} \text{ otherwise.} \end{cases} \quad (9)$$

Based on $w_i(k)$, the age function $h_i(k)$ evolves as follows:

$$h_i(k+1) = \begin{cases} 1 & \text{if } A(k) = i,\ Z_i(k) = 1 \\ w_i(k) + 1 & \text{if } A(k) = 3,\ Z_i(k) = 1 \\ h_i(k) + 1 & \text{otherwise.} \end{cases} \quad (10)$$

A sample path for the evolution of w_1, h_1 and w_2, h_2 is shown in Fig. 2. The initial state is $(h_1, h_2, w_1, w_2) = (3, 4, 1, 2)$, and the actions and the channels are as follows.

k	1	2	3	4	5
$A(k)$	2	1	1	2	3
$(Z_1(k), Z_2(k))$	(1,1)	(1,0)	(0,1)	(1,0)	(1,1)

V. Coded Randomized Policies

Consider a stationary randomized policy where each action is chosen with a fixed probability in each time slot, independent of the system's status. Denote by μ_i the probability of action $A(k) = i$, $i \in \{1, 2, 3\}$, $k \in \{1, \ldots, T\}$, where $\mu_1 + \mu_2 + \mu_3 = 1$.

We will first find the exact EWSAoI of the coded randomized policy. We start with $\Delta_i(m)$ derived in (3). Consider $i = 1$. The expectation of $\Delta_1(m)$ is

$$\mathbb{E}[\Delta_1(m)|\vec{h}_0] = \mathbb{E}\left[\frac{I_1^2(m)}{2} + D_1(m-1)I_1(m) - \frac{I_1(m)}{2}\Big|\vec{h}_0\right]$$
$$\overset{(a)}{=} \frac{\mathbb{E}\left[I_1^2\right]}{2} + \mathbb{E}[D_1]\mathbb{E}[I_1] - \frac{\mathbb{E}[I_1]}{2}.$$

Equality (a) holds because of the following observations: (i) the processes $\{I_1(m)\}_m$ and $\{D_1(m)\}_m$ are each iid and not dependent on $\vec{h}_0$ (so we use I_1 and D_1 to denote the underlying random variables, respectively), and (ii) $I_1(m)$ and $D_1(m-1)$ are independent (while $I_1(m)$ and $D_1(m)$ may be dependent). Thus, the EWSAoI of user 1 is

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\mathbb{E}[h_1(k)|\vec{h}_0]=\lim_{T\to\infty}\frac{1}{T}\sum_{m=1}^{N_1(T)}\mathbb{E}[\Delta_1(m)|\vec{h}_0] \tag{11}$$

$$=\lim_{T\to\infty}\frac{N_1(T)}{T}\left(\frac{\mathbb{E}\left[I_1^2\right]}{2}+\mathbb{E}[D_1]\mathbb{E}[I_1]-\frac{\mathbb{E}[I_1]}{2}\right) \tag{12}$$

$$\overset{(b)}{=}\frac{\mathbb{E}\left[I_1^2\right]}{2\mathbb{E}[I_1]}+\mathbb{E}[D_1]-\frac{1}{2}. \tag{13}$$

where (b) holds because the arrival process is a renewal process [13] and hence $\lim_{T\to\infty}\frac{T}{N_1(T)}=\mathbb{E}[I_1]$.

Then, we consider the statistics of D_1 and I_1. At each slot k, and for $d\geq 1$, we have

$$\begin{aligned}&P(h_1(k)=1)=\mu_1(1-\epsilon_1),\\&P(h_1(k)=d)=\mu_1\mu_3(1-\epsilon_1)(\epsilon_1-\epsilon_{12})\\&\qquad\times(\mu_1\epsilon_{12}+\mu_2+\mu_3\epsilon_1)^{d-2}\quad d\geq 2.\end{aligned} \tag{14}$$

To find the probability distribution of D_1, condition the above probabilities on the event that a packet is delivered to user 1 at time slot k. The probability of this event can be found by summing (14) over all $d\geq 1$:

$$P^1_{\text{delivery}}=\frac{\mu_1(1-\epsilon_1)(\mu_1+\mu_3)(1-\epsilon_{12})}{1-\mu_1\epsilon_{12}-\mu_2-\mu_3\epsilon_1}.$$

We thus find

$$P(D_1=d)=\begin{cases}\mu_1(1-\epsilon_1)/P^1_{\text{delivery}} & d=1\\ \frac{\mu_1\mu_3(\epsilon_1-\epsilon_{12})(1-\epsilon_1)(\mu_1\epsilon_{12}+\mu_2+\mu_3\epsilon_1)^{d-2}}{P^1_{\text{delivery}}} & d\geq 2\end{cases}$$

and the expectation of D_1 is equal to

$$E(D_1)=1+\frac{\mu_3(\epsilon_1-\epsilon_{12})}{(\mu_1+\mu_3)(1-\epsilon_{12})(1-\mu_1\epsilon_{12}-\mu_2-\mu_3\epsilon_1)}. \tag{15}$$

The distribution of I_1 can be found by treating I_1 and D_1 jointly. First of all, we have

$$P(I_1=1)=P(I_1=1,D_1=1)=\mu_1(1-\epsilon_1).$$

Then we look at the event of $I_1=\ell$ and $D_1=d$ for $\ell\geq 2, d\leq\ell$ (otherwise, if $d>\ell$, then $\Pr(I_1=\ell,D_1=d)=0$ because the packet in the queue $Q_2^{(1)}$ becomes obsolete once a new packet is delivered to user 1). So we suppose $d\leq\ell$, and consider the following cases: (i) If $d=1$, then a packet was delivered by action i at slot ℓ. (ii) For $d\geq 2$, the delivered packet was moved to $Q_2^{(1)}$ at slot $\ell-d+1$, stayed there, and got received at user 1 at slot ℓ. Now consider slots 1 to $\ell-d+1$. Denote by t the first slot in which a packet is received in $Q_2^{(1)}$, $1\leq t\leq\ell-d+1$. Then we have the two sub-cases: (ii-1) If t exists, then $Q_2^{(1)}$ is empty before t, and the delivered packet (another packet different from the delivered packet) moves to $Q_2^{(1)}$ at t when $t=\ell-d+1$ (when $t<\ell-d+1$) and from that point $Q_2^{(1)}$ is non-empty. (ii-2) If there is no such slot t, which may happen for $d=1$, then $Q_2^{(1)}$ is empty for the entire duration of ℓ slots. Considering the above cases, for $\ell=1,2,\ldots$, $d\leq\ell$, we can show the following lemma (see [12] for the proof).

Lemma 1. *The probability distribution of the inter delivery random variable I_1 is given by*

$$P(I_1=\ell)=\delta_1 x_1^{\ell-1}+\beta_1 y_1^{\ell-1} \tag{16}$$

where

$$x_1=\mu_1\epsilon_{12}+\mu_2+\mu_3$$

$$y_1=\mu_1\epsilon_1+\mu_2+\mu_3\epsilon_1$$

$$\delta_1=\mu_1(1-\epsilon_{12})+\frac{\mu_1^2(\epsilon_1-\epsilon_{12})(1-\epsilon_{12})}{-\mu_1(\epsilon_1-\epsilon_{12})+\mu_3(1-\epsilon_1)}$$

$$\beta_1=-\frac{\mu_1(\epsilon_1-\epsilon_{12})(1-\epsilon_1)}{-\mu_1(\epsilon_1-\epsilon_{12})+\mu_3(1-\epsilon_1)}\Big(\mu_1+\mu_3\Big).$$

Using Lemma 1, we find $\mathbb{E}[I_1]$ and $\mathbb{E}[I_1^2]$ to be:

$$\mathbb{E}[I_1]=\frac{\delta_1}{(1-x_1)^2}+\frac{\beta_1}{(1-y_1)^2} \tag{17}$$

$$\mathbb{E}[I_1^2]=\frac{\delta_1(1+x_1)}{(1-x_1)^3}+\frac{\beta_1(1+y_1)}{(1-y_1)^3}. \tag{18}$$

Finally, substituting (15), (17), (18) into (13), we find EWSAoI as given by the following theorem.

Theorem 2. *The $EWSAoI$ of Randomized policy is characterized by*

$$E[J^R]=\frac{1}{2}\sum_{i=1}^{2}\alpha_i\left(\frac{\frac{\mu_3(1-\epsilon_i)}{(\mu_i(1-\epsilon_{12}))^2}+\frac{-\mu_i(\epsilon_i-\epsilon_{12})}{((\mu_i+\mu_3)(1-\epsilon_i))^2}}{\frac{\mu_3(1-\epsilon_i)}{\mu_i(1-\epsilon_{12})}+\frac{-\mu_i(\epsilon_i-\epsilon_{12})}{(\mu_i+\mu_3)(1-\epsilon_i)}}+\frac{\mu_3(\epsilon_i-\epsilon_{12})}{(\mu_i+\mu_3)(1-\epsilon_{12})(\mu_i(1-\epsilon_{12})+\mu_3(1-\epsilon_i))}\right). \tag{19}$$

Remark 1. *To find an* optimal coded randomized policy *with respect to age, we have to choose the probability vector $(\mu_1^*,\mu_2^*,\mu_3^*)$ such that $E[J^R]$ is minimized.*

Remark 2. *Setting $\mu_3=0$, we recover the EWSAoI of [11] which corresponds to uncoded randomized policies.*

A. Symmetric BPECs

Consider the class of symmetric BPECs. Let the erasure probabilities of both channels to users 1 and 2 be equal to ϵ and ϵ_{12} be the probability of simultaneous erasure at both users. So $\epsilon>\epsilon_{12}$. Note that ϵ_{12} is either a function of ϵ or a constant, thus we rewrite ϵ_{12} as $\epsilon_{12}(\epsilon)$. For simplicity, let $\alpha_1=\alpha_2$. We find regimes of operation where optimal coded randomized policies strictly improve EWSAoI over uncoded randomized policies such as [11].

Theorem 3. *Optimal coded and uncoded randomized policies achieve the same age if and only if $\epsilon_{12}(\epsilon)-2\epsilon+1\geq 0$. Otherwise, optimal coded randomized policies strictly outperform uncoded randomized policies.*

Remark 3. *When the channels are independent, i.e.,* $\epsilon_{12}(\epsilon) = \epsilon^2$*, coded and uncoded randomized policies have the same performance with respect to age.*

VI. Max-Weight Policy

In this section, we devise deterministic policies using techniques from Lyapunov Optimization. Denote $\vec{s}(k) = (h_1(k), h_2(k), w_1(k), w_2(k))$. Define the Lyapunov function.

$$L(\vec{h}(k)) = \frac{1}{2}\sum_{i=1}^{2} \alpha_i h_i^2(k), \tag{20}$$

and the one-slot Lyapunov Drift

$$\Delta(\vec{h}(k)) = E[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{s}(k)]. \tag{21}$$

We devise the Max-Weight (MW) policy such that it minimizes the one-slot Lyapunov drift. In particular, in each slot k, the MW policy chooses the action that has the maximum weight as shown in following Table

$A(k)$	Weights
1	$\frac{\alpha_1(1-\epsilon_1)}{2} h_1(k)(h_1(k)+2)$
2	$\frac{\alpha_2(1-\epsilon_2)}{2} h_2(k)(h_2(k)+2)$
3	$\frac{1}{2}\sum_i \alpha_i(1-\epsilon_i)1_{\{w_i(k)>0\}}(h_i^2(k)+2h_i(k)-w_i^2(k)-2w_i(k))$

While the exact analysis of the resulting EWSAoI is difficult for the above MW policy, we derive an upper bound on it as stated below. The proof can be found in [12].

Theorem 4. *The EWSAoI achieved by the proposed Max-Weight policy is upper bounded by*

$$L_{UB}^{MW} = \sqrt{\frac{1}{2}\sum_{i=1}^{2} \frac{\alpha_i}{\mu_i(1-\epsilon_i)} \sum_{i=1}^{2} \alpha_i \Psi_i} + \frac{1}{2}\sum_{i=1}^{2} \alpha_i \Phi_i$$

where

$$\Phi_i = \frac{1 - \mu_3 P_{ne}^i(1-\epsilon_i) - \mu_i(1-\epsilon_i)}{\mu_i(1-\epsilon_i)},$$

$$\Psi_i = 1 - \mu_3 P_{ne}^i(1-\epsilon_i) + \frac{\left(1 - \mu_i(1-\epsilon_i) - \mu_3 P_{ne}^i(1-\epsilon_i)\right)^2}{\mu_i(1-\epsilon_i)},$$

$$P_{ne}^i = \frac{\mu_i(\epsilon_i - \epsilon_{12})}{\mu_3(1-\epsilon_i) + \mu_i(\epsilon_i - \epsilon_{12}) - \mu_3\mu_i(1-\epsilon_i)(\epsilon_i - \epsilon_{12})}.$$

VII. Numerical results and discussion

In this section, we compare the performance of the proposed coded algorithms with the uncoded algorithms in [11]. In Figure 3, we plot the EWSAoI of optimal coded and uncoded randomized policies, coded and uncoded Max-Weight policies, as well as the lower bound of Theorem 1. We have chosen $\alpha_1 = 0.3$, $\alpha_2 = 0.7$, and $\epsilon_1 = \epsilon_2 = \epsilon$, where ϵ varies from 0.5 to 0.9. We consider a dependent channel with $\epsilon_{12} = \epsilon^2/5$. We see that for the proposed policies, coding is beneficial when the channel erasure is larger than a threshold. Although the gain is small for MW policies, we believe the gain will be more significant over networks with many users.

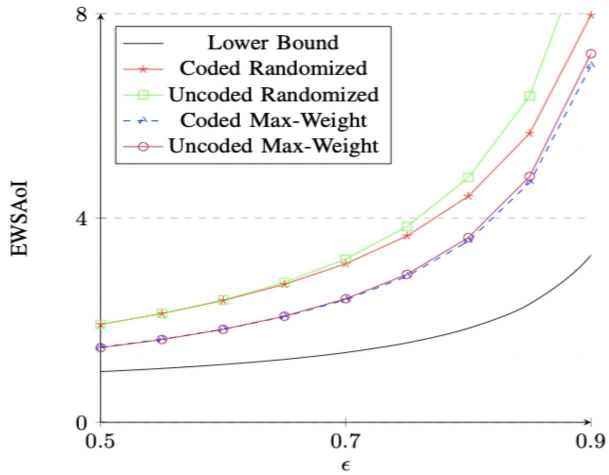

Fig. 3: EWSAoI as a function of erasure probability for a class of dependent channels with $\epsilon_{12} = \epsilon^2/5$.

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